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# Efficient Dynamic Pollution Taxation in an Uncertain Environment

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**Abstract.** This paper analyzes efficient pollution taxation within a stochastic model of endogenous growth. Pollution is a by-product of production and causes disutility. Furthermore, the productivity which results from environmental quality is uncertain. This reflects e.g. uncertain capital depreciation induced by natural disasters like hurricanes or floods. This uncertainty is shown to raise an ambiguous impact on the optimal pollution level as well as on optimal environmental taxation. Market equilibrium turns out to be suboptimal, since the households mis-perceive their individual impact on pollution. Conditions for welfare maximizing pollution taxation are stated and it is shown that a direct pollution tax is not appropriate to yield Pareto-optimal growth. Instead, a linear capital income tax together with a linear abatement subsidy build an efficient tax scheme, if secondarily the governmental budget is balanced. Moreover, an increase in the riskiness of environmental productivity may even lead to an increase in the optimal pollution level and to a decrease in optimal environmental taxation, depending predominantly on the preference parameters.

**Key words:** pollution, taxation, uncertainty, endogenous growth

**JEL classification:** D8, D9, H2, O1, O4, Q2

## 1. Introduction

The main question of this paper is how a society should react to environmental risk. Given that we do not know the detailed consequences of global warming on our future well being: should we aspire less carbon dioxide emissions due to the risk as a precaution? Should we increase the pollution tax as a response to risk? I develop a model of a growing economy subject to pollution externalities and uncertainty. The analysis of the model shows that there are no unambiguous answers to the questions raised above. When uncertainty about the link between pollution and production is bigger, the optimal level of pollution may become larger and the optimal pollution tax smaller. The former is due to the rise in optimal capital accumulation which can result from the increase in risk. The latter is caused by the increase in equilibrium abatement activity going back to the extended risk associated with pollution.

This paper discusses the dynamic situation with pollution as a by-product of production and with environmental risk: Environmental quality is assumed to have an impact on productivity, but this impact is uncertain. The number of employees away sick increases due to environmental degradation, but the number away sick in a specific firm is not known with certainty in advance. Environmental degradation also increases the destructive power of hurricanes or floods. Nevertheless, it is uncertain which firm will be affected by unforeseen depreciation. The main focus of the paper is to develop welfare maximizing pollution taxation in a dynamic and uncertain world. Many contributions analyze environmental policies within a static, riskless setting. I demonstrate that the optimality of environmental tax schemes does not necessarily carry over to the stochastic dynamic setting. In particular, a tax which is levied directly on pollution, is shown to be inconsistent with steady state growth. Additionally, as long as the considered risk is idiosyncratic, a welfare maximizing pollution tax provides an insurance against the income volatility caused by environmental risk and thereby completely eliminates income uncertainty. The pollution level reacts ambiguously to the abolition of risk, depending on the parameter setting.

Various contributions examine the effect of environmental degradation on endogenous growth, as e.g. Gradus and Smulders (1993), Ligthart and van der Ploeg (1994), Bovenberg and Smulders (1997), Jones and Manuelli (1995), Byrne (1997) or Stokey (1998). The authors derive conditions for the existence of sustainable growth paths and analyze environmental policy to internalize market failures. In general, the effect of environmental aspects on the growth process differs with respect to the underlying production structure. If environmental degradation is an inevitable by-product of the consumption good, as e g. in the approach of Stokey (1998), sustainable growth is unlikely due to the trade-off between consumption and environment. In contrast, if the engine of growth is independent from environment, as e.g. the accumulation of human capital in the Lucas-type model of Gradus and Smulders (1993) or Byrne (1997), the optimal (sustainable) growth path may even be unaffected by environmental concerns. Hartman and Kwon (2005) show in a related setting, that an environmental Kuznets curve might occur.

In the presence of pollution, uncertainty is important, since risk averse individuals will adjust their decisions to the underlying uncertainty. Although risk is a determining factor of the evolution of environmental quality as well as of the growth process, there are only few papers which address the impact of risk on environmental development, as e.g. Baranzini and Bourguignon (1995), Beltratti (1998), Chichilnisky and Heal (1998) as well as Ayong Le Kama and Schubert (2004) or Keller et al. (2004). Uncertainty gains importance for the dynamic macroeconomic equilibrium mainly through two different ways: On the one hand side, risk averse individuals react on the underlying risk within their intertemporal decision concerning consumption, abatement, and capital accumulation. The reaction is ambiguous and depends crucially on intertemporal substitution as well as on risk aversion (see e.g. Soretz 2003, 2004). On the other hand side, the impact of environmental policy changes due to uncertainty. Any governmental activity influences not only expected values of net economic variables, but also their volatility. This leads to counter acting effects on the equilibrium growth process, which were analyzed first in the seminal work of Eaton (1981) and more recently taken up e.g. by Turnovsky (1993, 1995b, 2000), Smith (1996a), Clemens and Soretz (1997, 2004), or Corsetti (1997).

The usual assumption of environmental quality as a pure public good is relaxed in this model. Instead, the agents take part of their influence on pollution into account within individual optimization: With respect to some aspects, environmental quality exhibits rivalry. For instance, vegetables which are cultivated without pesticides, are healthful for the particular consumer *and* for the whole society. The extent to which pollution is perceived to be unattached to individual decisions is parameterized according to the formulation of congestion effects in the public goods literature (see e.g. Edwards 1990, Glomm and Ravikumar 1994; Fisher and Turnovsky 1998; Turnovsky 1999). Nevertheless, due to this partial perception, market equilibrium is suboptimal, and gives the reason for pollution taxation.

The main focus of this paper is the following: In a stochastic environment which emphasizes the uncertainty of the productivity due to environmental quality, conditions for efficient pollution taxation are developed. It is shown that due to uncertainty, a tax which is levied directly on pollution, is not suitable to obtain socially optimal growth. However, a linear tax on capital together with a linear subsidy on abatement and a balanced governmental budget is a simple example for welfare maximizing pollution taxation. If the uncertainty associated with pollution is idiosyncratic, government is able to provide an insurance against the involved income volatility. Hence, a complete insurance results to be required for a first best pollution tax in a society of risk averse individuals. Nevertheless, the first best growth rate as well as the first best sustainable pollution level react ambiguously to the elimination of risk.

The assumptions of the model are presented in Section 2. Section 3 develops the Pareto-optimal growth path to serve as reference setting. Part 4 derives the decentralized equilibrium. Section 5 establishes conditions for efficient environmental policy and analyzes different types of pollution taxation. Section 6 gives a short conclusion.

## 2. The Model

According to the formulation of Smulders and Gradus (1996), environmental quality affects the economy through various channels: first, pollution is an inevitable by-product of production. Second, environmental quality affects the productivity within the consumption good sector. Third, pollution causes disutility. These three effects will be defined in the following.

In the underlying economy pollution is caused by capital accumulation and reduced by means of abatement effort. Hence, within the growth process, pollution increases as a by-product whenever the households invest in physical capital. Pollution decreases when households decide to raise abatement expenditures. Smulders and Gradus (1996) show that sustainable growth with non-increasing long-run pollution in this setting only is feasible if the elasticity of pollution with respect to abatement is greater or equal than the elasticity of pollution with respect to capital. Roughly speaking, pollution is non-increasing during the growth process, if abatement is at least as effective than capital with respect to pollution. Here, the limiting case with equal elasticities will be considered. Without loss of generality,[1] individually caused pollution, $P_i(t)$ is simply defined by the ratio between the individual capital stock, $k_i(t)$, and individual abatement effort, $e_i(t)$,

$$P_i(t) = P(k_i(t), e_i(t)) = \frac{k_i(t)}{e_i(t)} \tag{1}$$

such that the elasticities of pollution with respect to capital and abatement are equal (and unity).[2]

Furthermore, pollution is considered to be a flow variable, hence the model can predominantly be applied to pollutants which dissolve rather quickly. Nevertheless, this assumption seems maintainable since the pollution level is linked to the stock of physical capital. Therefore, ongoing capital accumulation ceteris paribus induces a perpetual increase in pollution.

The second effect of pollution refers to the productivity within the consumption good sector. The consumption good is produced by the only input factor capital and by means of a linear technology. In order to keep the framework as simple as possible, labor is neglected. Hence, $k_i(t)$ should be interpreted as broad measure of capital, including human capital. The production function of household $i$ follows Smulders and Gradus (1996) or Stokey (1998) and can be written as

$$f_i(k_i(t)) = Ap_i(t)k_i(t) \tag{2}$$

where productivity $A\, p_i(t)$ depends on environmental quality. An increase in environmental quality raises productivity e.g. by reducing depreciation of physical capital or by enhancing the health of workers (see Smulders and Gradus 1996, p. 508). Moreover, the productivity impact of the environment, $p_i(t)$, is not known with certainty in advance. A reason for this assumption is that uncertainty is a main feature of environmental degradation: Low environmental quality e.g. increases the power and the quantity of hurricanes which cause capital depreciation. Nevertheless, the occurrence of hurricanes is rather stochastic than deterministic. It is not known with certainty, when the next hurricane will occur and which factory it will destroy.

Therefore, only the expected value of environmental quality is known, but additionally there is environmental uncertainty: The productivity effect of environmental quality is stochastic and determined by[3]

$$p_i(t) = P(t)^{-\alpha}dt + P(t)^{\alpha'}\sigma dz_i(t) \ \alpha, \ \alpha' > 0 \tag{3}$$

where $dz_i \sim N(0,dt)$ denotes the individual specific increment to a Wiener process.[4] Since environmental quality is a public good, aggregate pollution, $P(t)$, is relevant for environmental quality and therefore determines the expected productivity effect of pollution. With the assumption of a continuum of individuals with homogenous preferences as well as homogenous technology, it will be derived subsequently, that all individuals emit the same amount of pollution. Additionally, population size is normalized to unity,[5] so the aggregate will be described by the average. Therefore, aggregation of the individual pollution levels ends up in the identity of aggregate (average) and individual pollution. With the definition given in (3), $\alpha$ determines the absolute value of the elasticity of expected production with respect to pollution. Hence $\alpha > 1$ ($0 < \alpha < 1$) means that an increase in pollution by 1 percent ceteris paribus results in a decrease in expected production by more (less) than one percent. This is equivalent to an increasing (decreasing) marginal productivity of environmental quality. The case $\alpha = 1$ indicates a constant marginal product of environmental quality. From the empirical point of view, none of these cases can be excluded. The value of $\alpha$ depends on the considered industry.

A natural disaster, e.g. the occurrence of a hurricane or a flood, is represented by a low (or negative) realization of the stochastic disturbance, $dz_i$, which results in a low productivity. With this definition, the standard deviation of environmental productivity, $P_i^{\alpha'}\sigma dt$, increases with the aggregate pollution level. This assumption reflects the fact that lower environmental quality comes along with an increase in the destructive power of hurricanes or an increase in the water level of floods. Therewith the absolute value of environmental degradation which results from natural disasters increases with the mean level of pollution used in production.[6] This assumption resembles the settings of Fernandez (2005) or Lafforgue (2005).

The features of the stochastic process, $dz_i$, (the expected value as well as the riskiness) are equal for all firms. Nevertheless, the realization of the environmental risk is individual for each firm. The reason is that this paper focuses on idiosyncratic environmental risk, as e.g. the number of staff away sick due to environmental reasons, rather than aggregate risk, which affects the whole society, as e.g. accidents in a nuclear power station. Of course, in reality most types of environmental risk are neither idiosyncratic nor aggregate. The probabilities that different factories will be destroyed by a hurricane rather are positively correlated. Nevertheless, hurricanes or floods usually emerge locally, hence in the following they are considered as approximately idiosyncratic. Moreover, the inclusion of correlated environmental risk to the point of aggregate risk does not imply major changes to the results.

The utility of individual $i$ depends on his consumption path, $c_i(t)$, as well as on the aggregate pollution path, $P(t)$. Furthermore, the individuals are infinitely long lived[7] and their intertemporal utility is defined according to the recursion

$$G((1-\rho)u_i(t)) = \frac{1-\rho}{1-1/\varepsilon}\left(c_i(t)P(t)^{-\gamma}\right)^{1-1/\varepsilon}h + \exp(-\beta h)G((1-\rho)\mathrm{E}_t[u_i(t+h)]) \tag{4}$$

$$\text{with } G_i = G(x_i) = \frac{1-\rho}{1-1/\varepsilon}x_i^{\frac{1-1/\varepsilon}{1-\rho}} \text{ and } \varepsilon \neq 1, \rho \neq 1. \tag{5}$$

The constant rate of time preference is denoted with $\beta > 0$, and $\gamma > 0$ indicates disutility out of pollution. With an increase in $\gamma$, environmental amenities gain importance.

Recursive preferences were applied to stochastic growth by Obstfeld (1994) and later for instance by Smith (1996b), in order to distinguish the effects of risk taking from those of intertemporal substitution. The recursive specification of intertemporal utility draws back on Epstein and Zin (1989) or Weil (1990) and was extended to continuous time by Svensson (1989) and Duffie and Epstein (1992). It allows for a constant intertemporal elasticity of substitution, $\varepsilon > 0$, as well as a constant degree of relative risk aversion, $\rho > 0$. Nevertheless, it is possible to set these two parameters separately. In the special case where $1/\varepsilon = \rho$, the recursive preferences (4) result in the usual expected utility form.

## 3. Pareto-optimal Growth

In this section, the socially optimal growth path will be determined. Therefore, a social planner chooses individual consumption and abatement expenditures together with capital accumulation in order to maximize lifetime utility, $u_i(t)$, as defined by the recursion (4) and (5), with respect to the capital accumulation process

$$dk_i = [Ak_iP^{-\alpha} - c_i - e_i]dt + Ak_iP^{\alpha'}\sigma dz_i. \tag{6}$$

The stochastic Bellman equation is derived by means of Itô's Lemma since the stochastic accumulation process of capital cannot be differentiated with respect to time. Additionally, let the value function, $J_i(k_i)$, denote maximum lifetime utility of individual $i$. The optimization problem of the social planner then results in the stochastic Bellman equation

$$\begin{aligned}\mathcal{B} = &\frac{1-\rho}{1-1/\varepsilon}(c_i P^{-\gamma})^{1-1/\varepsilon} - \beta G((1-\rho)J_i(k_i)) \\ &+ (1-\rho)G'((1-\rho)J_i(k_i))\left(J_i'(k_i)\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}J_i''(k_i)\sigma_{k_i}^2\right)\end{aligned} \tag{7}$$

with given initial values of physical capital, $k_{i0}$, and of the stochastic disturbance, $z_{i0} = 0 \ \forall \ i$. The variance of individual capital, $\sigma_{k_i}^2 \equiv (E[dk_i^2] - E[dk_i]^2)/dt = A^2 k_i^2 P^{2\alpha'}\sigma^2$, is determined by the volatility of the productivity effect of environmental quality.

Maximization of the Bellman equation with respect to consumption leads to the first necessary condition

$$c_i^{-1/\varepsilon} P^{-\gamma(1-1/\varepsilon)} - G_i' J_i' \stackrel{!}{=} 0 \tag{8}$$

which balances marginal utility out of consumption across time. The assumptions of constant intertemporal elasticity of substitution, $\varepsilon$, and constant degree of relative risk aversion, $\rho$, together with an intratemporal elasticity between consumption and pollution which is unity lead to the following conjecture

$$\begin{aligned}\mu_i &\equiv \frac{c_i}{k_i} = \mu \quad \forall i, t \\ \eta_i &\equiv \frac{e_i}{k_i} = \eta \quad \forall i, t\end{aligned} \tag{9}$$

of constant consumption and abatement ratios. Particularly, this means the independence of consumption and abatement ratios from the individual capital stock and implies homogeneity of all households with respect to these decisions, independent from the realization of their environmental productivity time path. Hence, consumption and abatement activities are presumed to grow at the same rate as physical capital in the socially optimal steady-state and this growth rate is supposed to be equal for all individuals. Then neither the distribution of initial endowment nor the time path of environmental productivity have an impact on expected growth. Growth as well as the optimal ratios (9) are equal for all individuals. Moreover, since population is normalized to unity, aggregate economic variables are identical with their mean values.

Together with the definition (5) of $G_i$, substitution of $P = \eta^{-1}$, and the necessary condition (8), this conjecture results in a CRRA guess for the value function

$$J(k_i) = \left(\mu\eta^{\gamma(1-\varepsilon)}\right)^{\frac{1-\rho}{1-\varepsilon}} \frac{k_i^{1-\rho}}{1-\rho}. \tag{10}$$

Maximization of the Bellman equation with respect to abatement expenditures, $e_i$, gives the second necessary condition

$$\gamma(c_i P^{-\gamma})^{1-1/\varepsilon} + G_i' \left( J_i'(\alpha A k_i P^{-\alpha} - e_i) + \frac{1}{2} J_i'' e_i \frac{\partial \sigma_{k_i}^2}{\partial e_i} \right) \stackrel{!}{=} 0. \tag{11}$$

which balances marginal utility out of consumption and abatement, and yields intratemporal efficiency. Substitution of the value function (10), the definition (5) of $G_i$ and the constant ratios μ and $\eta$ leads to the following relationship

$$\frac{\mu\gamma}{\eta} = 1 - \alpha A \eta^{\alpha-1} \left( 1 + \rho \frac{\alpha'}{\alpha} A \eta^{-2\alpha'-\alpha} \sigma^2 \right). \tag{12}$$

which equates the marginal rate of (intratemporal) substitution, $dc/de$, with the price relation, and determines optimal consumption and abatement expenditures. The right hand side of Equation (12) gives the certainty equivalent of the marginal cost of an increase in abatement (expressed in units of the consumption good): first, there is a direct cost of one unit, and second, there is an indirect negative cost since environmental quality raises the productivity of the consumption good sector.[8] The certainty equivalent of the marginal cost of an increase in abatement, expressed in units of the consumption good, is equated with the marginal rate of substitution, $dc/de$, which is given on the left hand side of Equation (12). Hence, the negative impact of pollution on output (−α) ceteris paribus increases the optimal abatement ratio, compared with a situation where pollution had no effect on production (α = 0). This replicates the additional benefit of abatement activity due to enhanced productivity.

Furthermore, uncertainty (positive $\sigma^2$) ceteris paribus decreases the optimal abatement ratio. With more abatement activity, pollution decreases and therefore the volatility of environmental productivity diminishes: if a cleaner production process is chosen, there is less risk of natural disasters which reduce productivity. The more risk averse the households are (higher $\rho$), the stronger is the socially optimal reaction on environmental risk. Nevertheless, the entire impact of uncertainty on the socially optimal pollution level is more complex and turns out to be ambiguous. The relationship (12) contains the optimal consumption ratio, μ, as well as the optimal abatement ratio, $\eta$.

Both will be determined subsequently together with Equation (14) and react ambiguously on environmental risk, as explained in more detail with respect to the optimal growth rate (16).

Maximization of the stochastic Bellman Equation (7) with respect to capital leads to the third necessary condition

$$
\begin{aligned}
&-\gamma(cP^{-\gamma})^{1-1/\varepsilon}k_i^{-1} + (1-\rho)G_i''J_i'\left(J_i'\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}J_i''\sigma_{k_i}^2\right) \\
&\quad + G_i'\left(J_i'(AP^{-\alpha}(1-\alpha)-\beta) + J_i''\left(\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}\frac{\partial\sigma_{k_i}^2}{\partial k_i}\right) + \frac{1}{2}J_i'''\sigma_{k_i}^2\right) \stackrel{!}{=} 0
\end{aligned} \tag{13}
$$

which determines optimal capital accumulation and weighs momentary utility out of consumption or abatement against future utility out of capital investment. Again, substitution of (5), (10) and (9) together with the optimal abatement decision (12) results in the optimal consumption ratio

$$
\mu = \varepsilon\beta + (1-\varepsilon)\left(A\eta^{\alpha} - \eta - \frac{\rho}{2}A^2\eta^{-2\alpha'}\sigma^2\right). \tag{14}
$$

An increase in capital accumulation yields the expected social return $A\eta^{\alpha} - \eta$, and simultaneously increases the riskiness of future income. Whether a rise in capital returns leads to a positive or a negative effect on consumption, depends on the elasticity of intertemporal substitution, $\varepsilon$, to be less or greater than unity. Together with relation (12), the socially optimal consumption and abatement ratios are determined. Condition (14) focuses on the dynamic trade-off between momentaneous consumption (or abatement) and capital accumulation, whereas Equation (12) describes the intratemporal trade-off between consumption and abatement.

Unfortunately, it is not straightforward to determine a closed-form solution. Inserting Equation (14) into (12) leads to the polynomial

$$
\begin{aligned}
&1-\alpha A\eta^{\alpha-1}\left(1+\rho\frac{\alpha'}{\alpha}A\eta^{-2\alpha'-\alpha}\sigma^2\right) \\
&\quad = \frac{\gamma}{\eta}\left(\varepsilon\beta + (1-\varepsilon)\left(A\eta^{\alpha} - \eta - \frac{\rho}{2}A^2\eta^{-2\alpha'}\sigma^2\right)\right).
\end{aligned} \tag{15}
$$

determining optimal abatement activity. Without further specification, it is impossible to calculate the optimal level of abatement expenditure.[9] Following, the solution will be based on the interpretation of condition (15). This condition equates marginal cost of abatement (left hand side) with marginal benefit of abatement (right hand side). The curves of marginal cost and marginal benefit can be shown to intersect uniquely[10] as long as the elasticity of intertemporal substitution is sufficiently low, $\varepsilon < 1$, which is the empirically relevant range (see e.g. Hall 1988; Epstein and Zin 1991). Figure 1 demonstrates the unique optimal solution for abatement effort.

Now, from goods market clearing, $dk = f - c - e$, it is straightforward to determine the expected capital growth rate

$$\varphi \equiv \frac{E[dk]}{kdt} = \varepsilon(A\eta^{\alpha} - \eta - \beta) + (1 - \varepsilon)\frac{\rho}{2}A^2\eta^{-2\alpha'}\sigma^2 \tag{16}$$

which describes socially optimal growth, given the optimal abatement ratio from Equation (15). Optimal expected growth, as well as optimal consumption and abatement indeed are independent from the capital stock and hence confirm the conjecture of equal and constant growth rates of all economic variables. Through optimal abatement effort, $\eta$, the social marginal return on capital is reduced and hence optimal expected growth falls short of the growth rate obtained with an $AK$-technology without pollution.

Furthermore, one can see that environmental risk affects socially optimal growth in an ambiguous way: First, due to the riskiness of environmental productivity, the certainty equivalent of capital return (unambiguously) decreases. Now, as well known, the optimal reaction on this decrease in the capital return depends on the elasticity of intertemporal substitution. If it is sufficiently low ($\varepsilon < 1$), the income effect dominates and leads to an increase in savings in order to compensate for the reduction in capital return. If instead the elasticity of intertemporal substitution is sufficiently high ($\varepsilon > 1$), optimal capital accumulation is reduced in favor of momentaneous consumption and abatement, due to the dominance of the substitution effect.[11]

Note that sustainable growth is feasible in the underlying economy: If social capital productivity is sufficiently high to allow for a positive growth rate in any time increment, expected growth will continue for all time increments. Additionally, optimal pollution remains constant since optimal

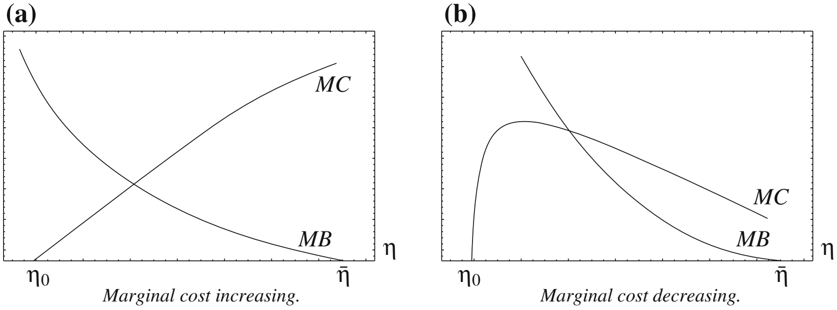



*Figure 1.* Optimal abatement effort.

abatement grows with the same (stochastic) rate as capital. Nevertheless, the impact of the environmental productivity risk on the optimal pollution level and the optimal growth path is not clear cut: The optimal response on an increase in the riskiness may consist of an increase in the pollution level together with an increase in growth, due to the dominance of the income effects.

Last, the transversality condition

$$\lim_{t\to\infty} \mathrm{E}[\exp(-\beta t)G((1-\rho)J(k_i))] = 0. \tag{17}$$

must be satisfied in order to assure the boundedness of the optimal solution.[12]

## 4. Dynamic Market Equilibrium

One major problem of environmental degradation is that usually there are externalities which lead to suboptimal outcomes of market solutions. In the following two sections, the goal is to describe optimal pollution taxation in the stochastic dynamic setting. Therefore, this section will demonstrate the specific failures of market equilibrium: There is a gap between private and social marginal costs of pollution. Households – being firms at the same time – underestimate their individual impact on aggregate pollution. It will be shown that in this setting only the intratemporal decision between consumption and abatement is disturbed, whereas the intertemporal decision about capital accumulation corresponds to the Pareto-optimum.

The environment is often assumed to be a public good. Hence, individuals free ride and do not take into account their individual impact on pollution at all. For two reasons, this assumption will be relaxed in the further analysis. First, there are various environmental goods, which display rivalry to some degree: for example, food or clothes which are ecologically compatible, on the one hand side protect the individual health (only of the individual who bought them), and on the other hand side protect the environment (of the whole society). Second, persons show to some degree morality with respect to the environment: Sometimes, people go by bike to the bakery (and not by car), because they know that this behavior reduces air pollution for the whole society, and for themselves, too. Accordingly, Eriksson (2004, p. 281) states that households are "... willing to pay an extra premium for a product if it were green".

Hence, the perception of the individual influence on pollution is parameterized: I do not assume a pure pollution externality, where individuals do not consider pollution at all. But the individuals neither feel completely responsible for their individual impact on pollution. In fact, they perceive pollution to depend in part, $\delta$, on the decisions of "the others", hence on aggregate (average) capital accumulation, $k$, and aggregate (average) abatement effort, $e$, which are exogenous to individual decisions. Only the (maybe very small) part $1-\delta$ of pollution is perceived to depend on "the own" behavior, hence on individual capital accumulation, $k_i$, and individual abatement expenditures, $e_i$. Perceived pollution, $P_\mathrm{p}$, is then given by

$$P_\mathrm{p} = \left(\frac{\mathrm{k(t)}}{e(t)}\right)^{\delta} \left(\frac{k_i(t)}{e_i(t)}\right)^{1-\delta} \delta \in [0,1). \tag{18}$$

and replaces pollution within utility and production functions for individual optimization.

This setting of perception relies on the formulation of congestion effects in the public goods literature (see e.g. Edwards 1990; Glomm and Ravikumar 1994; Turnovsky 1999). In these lines, $1-\delta$ can be interpreted as degree of locality of pollution. The part $\delta$ of pollution is global and affects the whole society in the same way. The part $1-\delta$ of pollution is local and affects only the individual which caused the pollution. With this respect, $1-\delta$ is the joint degree of rivalry of capital and abatement in the "production" of pollution (see Turnovsky 1995a, p. 405).

As long as the perception parameter is above zero, the agents underestimate their individual influence on pollution. The result is a negative externality of capital accumulation and a positive externality of abatement effort.

Subsequently will be shown that in market equilibrium the abatement ratios are equal for all individuals. Hence, the aggregate relation $k/e$ and the individual relation $k_i/e_i$ in perceived pollution (18) will be equal in equilibrium. Nevertheless, there is a continuum of individuals and the interaction between them is characterized by perfect competition. Therefore, they consider aggregate capital as well as aggregate abatement as exogenous within individual optimization.

Hence, the consumption and abatement ratios in market equilibrium have to fulfill[13]

$$1-(1-\delta)\alpha A\eta_\mathrm{M}^{\alpha-1}\left(1+\rho\frac{\alpha'}{\alpha}A\eta_\mathrm{M}^{-2\alpha'-\alpha}\sigma^2\right) = \\ = \frac{\gamma(1-\delta)}{\eta_\mathrm{M}}\left(\varepsilon\beta+(1-\varepsilon)\left(A\eta_\mathrm{M}^{\alpha}-\eta_\mathrm{M}-\frac{\rho}{2}A^2\eta_\mathrm{M}^{-2\alpha'}\sigma^2\right)\right) \tag{19}$$

In comparison with the corresponding condition (15) for Pareto-optimal abatement effort, marginal cost (given on the left hand sides of 15 and 19) increases and marginal benefit decreases due to mis-perception, $1-\delta$, as can be seen in Figure 2. Therefore, individually optimal abatement is reduced. This displays the externality due to partial perception of the individual impact on pollution. The households take only the part $1-\delta$ of their

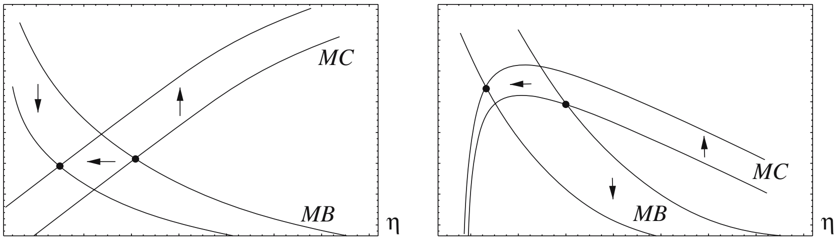



*Figure 2.* Impact of mis-perception on market equilibrium.

individual influence on pollution into account. Hence, they only perceive part of their disutility out of pollution ($\gamma\mu_M$) as well as part of the marginal environmental productivity ($\alpha A \eta_M^{-\alpha-1}$). Note, that individuals also take only part of their impact on environmental productivity risk into account. Hence, the positive effect of pollution uncertainty on equilibrium abatement effort is less than the effect on Pareto-optimal abatement.

The marginal benefit of abatement, given on the right hand side of Equation (19) is derived from the intertemporal decision between momentaneous consumption or abatement and future consumption through capital accumulation. This condition exactly replicates the corresponding condition for Pareto-optimal consumption. Since momentaneous consumption and future consumption are mis-perceived in the same way, there is no distortionary effect of partial perception on the intertemporal decision about capital accumulation.[14] Calculation of expected equilibrium growth out of market clearing $dk = f - c - e$ again displays this result

$$\varphi_M = \varepsilon\left(A\eta_M^{\alpha} - \eta_M - \beta\right) + (1-\varepsilon)\frac{\rho}{2}A^2\eta_M^{-2\alpha'}\sigma^2. \tag{20}$$

Note, that the conformity of intertemporal choice (20) with Pareto-optimal capital accumulation (16) does not imply that the expected growth rate in market equilibrium is Pareto-optimal. Due to partial perception, the equilibrium abatement ratio is suboptimally low, as already shown with Figure 2. Therefore, expected growth in market equilibrium is suboptimal. It is now straightforward to show that equilibrium expected growth is too high. The growth rate increases with a decline in the abatement ratio

$$\frac{\partial\varphi_M}{\partial\eta_M} = -\left(1 - \varepsilon\alpha A\eta_M^{\alpha-1}\left(1 + \rho\frac{\alpha'}{\alpha}A\eta_M^{-2\alpha'-\alpha}\sigma^2\right)\right) - \alpha'\rho A^2\eta_M^{-2\alpha'-1}\sigma^2 < 0. \tag{21}$$

With a suboptimally low equilibrium abatement ratio (due to the positive externality of abatement effort), the equilibrium growth rate results to be suboptimally high (due to the negative externality of capital accumulation).

This result gives rise to the introduction of pollution taxation: By means of a pollution tax, the true benefits and costs of pollution can be carried over to the households and market equilibrium can be improved.

## 5. Efficient Pollution Taxation

In order to internalize the externalities, a pollution tax will be incorporated. It will be demonstrated that with efficient pollution taxation, it is possible to realize Pareto-optimal equilibrium growth. Nevertheless, efficient pollution taxation implies distinct tax rates on capital and abatement on the one hand side, and different tax rates on deterministic and stochastic income components, on the other hand side.

The starting point is a generally formulated pollution tax

$$T_i(t) = T^{\mathrm{d}}(k_i(t), e_i(t))dt + T^{\mathrm{s}}(k_i(t), e_i(t))\sigma dz_i \tag{22}$$

which allows for a differentiated treatment of capital and abatement, as well as of deterministic and stochastic income parts, and will be determined efficiently in equilibrium. With this pollution tax, the evolution of the capital stock becomes

$$dk_i = [Ak_i P_{\mathrm{p}}^{-\alpha} - c_i - e_i - T^{\mathrm{d}}(k_i, e_i)]dt + (Ak_i P_{\mathrm{p}}^{\alpha'} - T^{\mathrm{s}}(k_i, e_i))\sigma dz_i \tag{23}$$

and the variance of capital is given by

$$\sigma_{k_i}^2 = (A^2 k_i^2 P_{\mathrm{p}}^{2\alpha'} - 2Ak_i P_{\mathrm{p}}^{\alpha'} T^{\mathrm{s}} + (T^{\mathrm{s}})^2)\sigma^2. \tag{24}$$

As subsequently will be shown, any optimal governmental policy requires a balanced governmental budget where capital tax revenues equate abatement subsidy payments. Therefore, it is most convenient to focus only on pollution taxation as given in Equation (22) and to neglect further governmental expenditures.

Macroeconomic equilibrium depends twofold on pollution taxation: First, the intratemporal decision between consumption and abatement is influenced by the pollution tax. Second, within the intertemporal savings decision of risk averse individuals, they react on the environmental policy. The influence of pollution taxation on the equilibrium abatement ratios results in

$$\frac{1}{1-\delta}\left(1 + T_{e_i}^{\mathrm{d}} - \alpha(1-\delta)A\eta_{\mathrm{T}}^{\alpha-1} - \rho\sigma^2\left(\alpha'(1-\delta)A\eta_{\mathrm{T}}^{-\alpha'-1} - T_{k_i}^{\mathrm{s}}\right)\right.$$
$$\left.\times\left(A\eta_{\mathrm{T}}^{-\alpha'} - \frac{T^{\mathrm{s}}}{k_i}\right)\right)$$

$$= \frac{\gamma}{\eta_{\mathrm{T}}}\left(\varepsilon\beta + (1-\varepsilon)(A\eta_{\mathrm{T}}^{\alpha} - \eta_{\mathrm{T}}) - \left(\frac{T^{\mathrm{d}}}{k_i} - \varepsilon\left(T^{\mathrm{d}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{d}}_{e_i}\right)\right)\right.$$
$$+ \frac{\rho}{2}\sigma^2\left((\varepsilon - 1)A^2\eta_{\mathrm{T}}^{-2\alpha'} + 2A\eta_{\mathrm{T}}^{-\alpha'}\left(\frac{T^{\mathrm{s}}}{k_i} - \varepsilon(T^{\mathrm{s}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{s}}_{e_i})\right)\right.$$
$$\left.\left. - \left((1+\varepsilon)\frac{T^{\mathrm{s}}}{k_i} - 2\varepsilon(T^{\mathrm{s}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{s}}_{e_i})\right)\frac{T^{\mathrm{s}}}{k_i}\right)\right) \tag{25}$$

Different from mis-perception, $\delta$, the pollution taxation acts on the marginal cost of abatement (left hand side) and on the marginal benefit (right hand side) manifold. As long as the pollution tax is not established concretely, the impact on macroeconomic equilibrium is not clear cut. Nevertheless, it is now possible to determine equilibrium growth

$$\varphi_{\mathrm{T}} = \varepsilon\left(A\eta_{\mathrm{T}}^{\alpha} - T^{\mathrm{d}}_{k_i} - \eta_{\mathrm{T}}(1 + T^{\mathrm{d}}_{e_i}) - \beta\right)$$
$$- \frac{\rho}{2}\sigma^2\left((\varepsilon - 1)A^2\eta_{\mathrm{T}}^{-2\alpha'} + 2A\eta_{\mathrm{T}}^{-\alpha'}\left(\frac{T^{\mathrm{s}}}{k_i} - \varepsilon(T^{\mathrm{s}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{s}}_{e_i})\right)\right.$$
$$\left. - \left((1+\varepsilon)\frac{T^{\mathrm{s}}}{k_i} - 2\varepsilon(T^{\mathrm{s}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{s}}_{e_i})\right)\frac{T^{\mathrm{s}}}{k_i}\right). \tag{26}$$

Taxation has both a direct and an indirect impact on equilibrium capital accumulation. The direct effect can be seen in Equation (26). The indirect impact is due to the adjustment of abatement effort. Within the direct influence, the pollution tax can again be shown to affect equilibrium growth through various channels: First, the expected returns on capital and abatement change. This can be seen in the first parenthesis of the growth rate (26). Expected capital return decreases due to the introduction of the pollution tax ($T^{\mathrm{d}}_{k_i}$) and leads to the usual growth diminishing effect of distortionary capital income taxation. A common pollution tax will decrease in abatement effort ($\eta_{\mathrm{T}}\ T^{\mathrm{d}}_{e_i} < 0$) and in this case foster equilibrium growth. Expected return on abatement increases and therefore facilitates capital accumulation.

Second, the stochastic pollution tax affects the uncertainty associated with environmental quality. The reaction of risk averse individuals on this change in future income risk can be analyzed with the second part of the growth rate (26). The impact of the stochastic pollution tax on equilibrium growth depends on the intertemporal elasticity of substitution, as this elasticity decides upon the dominance of income or substitution effects. Any positive stochastic pollution tax $T^{\mathrm{s}}$ will reduce the volatility of capital return as can be seen from the variance of capital (24). Actually, environmental taxation can provide a complete insurance against the income risk caused by environmental risk if the stochastic pollution tax takes the form $T^{\mathrm{s}}_i = \ A\ k_i\ \eta_{\mathrm{T}}^{-\alpha'}$. In this case, the stochastic pollution tax entirely offsets the income uncertainty caused by environmental productivity risk. Hence, the individuals are back in a world with a sure income flow ($\sigma^2_{k_i} = 0$) and expected equilibrium growth is given by

$$\varphi = \varepsilon\Big(A\eta_{\mathrm{T}}^{\alpha} - T^{\mathrm{d}}_{k_i} - \eta_{\mathrm{T}}(1 + T^{\mathrm{d}}_{e_i}) - \beta\Big) \tag{27}$$

and will be analyzed below.
Which attributes characterize the *optimal* pollution tax? Optimal pollution taxation has to adjust equilibrium economic variables to their Pareto-optimal levels. Welfare is determined by the propensity to consume out of wealth, $\mu$, and the abatement ratio, $\eta$, as can be seen from Equation (10). Both ratios deviate from their optimal levels. In particular, optimal taxation has to foster individual abatement effort, and to reduce equilibrium consumption. At the same time, equilibrium capital accumulation should not be affected directly, but only through the adjustment of the abatement ratio.

In the model considered here, we find a market failure due to mis-perception with the related consequences for consumption and abatement. Additionally, we find the institutional failure that the idiosyncratic environmental risk is not diversified neither by insurance nor by the government. This institutional failure exists by assumption, but can be justified with the moral hazard argument if the individual productivity risk is unobservable: If the insurance cannot distinguish between low income caused by low individual effort and low income due to a natural disaster, there is no incentive for abatement expenditures.

With respect to efficient pollution taxation, the analysis will be divided into two steps: first, tax schemes are analyzed which correct for the market failure due to mis-perception. Meanwhile, the uncertainty caused by environmental productivity will be regarded as exogenous. In the second step, a tax scheme is analyzed which corrects for the idiosyncratic environmental risk. Of course, this tax scheme can only be realized, if the government is capable to observe the individual realizations of environmental productivity.

### 5.1. DIRECT POLLUTION TAXATION

There is a negative externality of capital accumulation and a positive externality of abatement activity. Since both are caused by the same mis-perception of the individual influence on pollution (the same $\delta$), a direct pollution tax is suggesting in order to internalize the pollution externality. Just as well as in the deterministic case, it ought to be appropriate to close the gap between individually perceived and socially relevant impact on pollution. Nevertheless, the above stated conditions for optimal pollution taxation cannot be met by a tax, $T^{\mathrm{d}}(P)$, $T^{\mathrm{s}}(P)$, which is levied on pollution directly. In this case, the derivatives of the pollution tax with respect to capital and abatement would result in[15]

$$T^{\mathrm{d}}_{k_i} = (1-\delta)T^{\mathrm{d}\prime}(\eta_{\mathrm{T}}k_i)^{-1} \quad T^{\mathrm{s}}_{k_i} = (1-\delta)T^{\mathrm{s}\prime}(\eta_{\mathrm{T}}k_i)^{-1} \tag{28}$$

$$T^{\mathrm{d}}_{e_i} = -(1-\delta)T^{\mathrm{d}\prime}(\eta_{\mathrm{T}}e_i)^{-1} \quad T^{\mathrm{s}}_{e_i} = -(1-\delta)T^{\mathrm{s}\prime}(\eta_{\mathrm{T}}e_i)^{-1} \tag{29}$$

Indeed, in a deterministic growth model,[16] this kind of pollution tax – evaluated optimally – leads to the socially optimal steady state (see Smulders and Gradus 1996). The pollution tax has to be determined to equate decentral abatement, $\eta_{\mathrm{T}}$, as given by (25) and optimal abatement, $\eta$, as given by (15). Due to $T^{\mathrm{d}}_{k_i} + \eta_{\mathrm{T}}\ T^{\mathrm{d}}_{e_i} = 0$, the direct pollution taxation implies growth neutrality in the deterministic growth model, as required.

Nevertheless, this result does not extend to the stochastic dynamic setting. Since pollution is constant in any equilibrium, the deterministic as well as the stochastic part of the tax revenue are constant, too. With respect to the stochastic pollution tax, this rules out the possibility of steady state growth. In particular, the growth rate with direct pollution taxation, $T(P)$, becomes

$$\varphi_{T(P)} = \varepsilon(A\eta^{\alpha} - \eta - \beta) - \frac{\rho}{2}\sigma^2\left((\varepsilon - 1)A^2\eta^{-2\alpha'} + 2A\eta^{-\alpha'}\frac{T^{\mathrm{s}}}{k_i} - (1+\varepsilon)\left(\frac{T^{\mathrm{s}}}{k_i}\right)^2\right) \tag{30}$$

and hence is not independent from the level of capital accumulation. In order to enable steady state growth, the stochastic part of the pollution tax has to increase in capital. Only in this case, the volatility of the pollution tax increases in accordance with the volatility of the income stream induced by uncertain environmental productivity. If instead the stochastic pollution tax is constant, the riskiness of the pollution tax diminishes in relation to the riskiness of income, and steady state growth is impossible.

### 5.2. LINEAR TAXATION

Since all variables grow with a common rate, the relations between the economic variables remain constant on the steady state growth path. Hence, a simple efficient pollution tax consists in a linear tax on capital income ($\tau^{\mathrm{d}}_k$,$\tau^{\mathrm{s}}_k$) combined with a linear subsidy on abatement effort ($\tau^{\mathrm{d}}_e$,$\tau^{\mathrm{s}}_e$) and with balanced governmental budget

$$T^{\mathrm{d}*} = \tau^{\mathrm{d}}_k k_i + \tau^{\mathrm{d}}_e e_i = 0 \quad T^{\mathrm{s}*} = \tau^{\mathrm{s}}_k k_i + \tau^{\mathrm{s}}_e e_i = 0. \tag{31}$$

This implies growth neutrality due to

$$T^{\mathrm{d}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{d}}_{e_i} = 0 \quad T^{\mathrm{s}}_{k_i} + \eta_{\mathrm{T}} T^{\mathrm{s}}_{e_i} = 0. \tag{32}$$

and

$$T^{\mathrm{d}} = 0 \quad T^{\mathrm{s}} = 0. \tag{33}$$

The corresponding equilibrium expected growth rate hence coincides with the Pareto-optimal growth rate (16).

In order to adjust the abatement ratio, the optimal levels of the constant tax rates, $\tau^{\mathrm{d}}_k$, $\tau^{\mathrm{d}}_e$, $\tau^{\mathrm{s}}_k$, $\tau^{\mathrm{s}}_e$, become relevant. The balanced governmental budget implies

$$\tau^{\mathrm{d}}_k = -\eta_{\mathrm{T}} \tau^{\mathrm{d}}_e \quad \tau^{\mathrm{s}}_k = -\eta_{\mathrm{T}} \tau^{\mathrm{s}}_e. \tag{34}$$

The optimal solutions for the tax rates are evident from the equalization of equilibrium abatement, $\eta_{\mathrm{T}}$, and optimal abatement, $\eta$

$$\tau^{\mathrm{d}*}_k = \delta(\gamma\mu + \alpha A \eta^{\alpha}) \quad \tau^{\mathrm{s}*}_k = -\delta\alpha' A \eta^{-\alpha'}. \tag{35}$$

$$\tau^{\mathrm{d}*}_e = -\frac{\delta}{\eta}(\gamma\mu + \alpha A \eta^{\alpha}) \quad \tau^{\mathrm{s}*}_e = \frac{\delta}{\eta}\alpha' A \eta^{-\alpha'}. \tag{36}$$

The deterministic part of the pollution tax accounts for the pollution externalities in consumption as well as in production. The larger the parameter $\delta$, the more influential is the individual mis-perception. Individuals perceive a smaller part of their influence on pollution. Hence, there is more need for internalization, and capital income taxation as well as abatement subsidy rise (in absolute value). Note, that the tax on the stochastic component of capital income in fact is a subsidy ($\tau^{\mathrm{s}}_k{}^* < 0$). The reason is that households perceive only part of the riskiness of environmental productivity to depend on their individual decisions. The subsidy on the stochastic component of capital returns closes the gap between individually perceived and socially relevant environmental risk. With respect to the subsidy on abatement, the argument reverses: The stochastic part of abatement expenditure is taxed in order to reduce the incorporated uncertainty and to increase the attractiveness of abatement for risk averse individuals.

Moreover, the deterministic part of the pollution tax is affected by environmental risk via the optimal consumption and abatement ratios, $\eta$ and $\mu$. The impact of uncertainty is ambiguous, as already illustrated in Section 3. In the special case $\alpha = 0$ which neglects the property of environmental quality as a determinant of productivity and focuses on the disutility out of pollution,[17] the impact of uncertainty on the optimal deterministic pollution tax is given by

$$\frac{\partial \tau_k^{d*}}{\partial \sigma^2} = \delta\gamma \frac{\partial \mu}{\partial \sigma^2} = -\frac{\delta\gamma\rho A^2(1-\varepsilon)}{2(1+\gamma(1-\varepsilon))} < 0 \text{ for } \varepsilon < 1. \tag{37}$$

With an elasticity of substitution, $\varepsilon$, which is below unity, optimal growth and optimal pollution increase in reaction on environmental risk. Hence, there is less need for growth reduction by means of income taxation.

### 5.3. INSURANCE AGAINST INCOME RISK

Up to now, efficient pollution taxation was defined to correct for the market failure due to mis-perception. Uncertainty associated with pollution was considered arbitrarily. As long as environmental productivity risk is aggregate, or realization is not observable, there is an institutional failure which impedes any insurance and indeed prevents government from risk-pooling. In the following, we will relax this assumption and analyze a tax scheme which pools the income uncertainty associated to pollution by means of income taxation. Hence, welfare can even be enhanced, beyond the situation with arbitrarily given risk.

Whether the environmental risk is idiosyncratic or aggregate, depends on the specific case. Both occurs in reality. Examples may be staff away sick due to environmental reasons or a hurricane which emerges locally. This is the type of risk which is captured by this model with idiosyncratic environmental productivity. In contrast, aggregate environmental risk applies to the whole society: all firms experience the same realization of the productivity shock. An example for aggregate environmental risk is an accident in a nuclear power station.

For idiosyncratic environmental risk, the aggregate income tax revenue out of stochastic pollution taxation is zero. Nevertheless, on the individual level, the volatility of future income streams decreases with a rise in the stochastic pollution tax. This insurance argument of income taxation draws back on Domar and Musgrave (1944) and was further developed by Stiglitz (1969). The consequences of taxation on growth and welfare are studied e.g. by Smith (1996b) or Turnovsky (2000) within stochastic growth models. As already discussed with Equation (27), the stochastic pollution tax which entirely absorbs the uncertainty associated with pollution, is given by

$$\tau_k^s = A\eta^{-\alpha'} \quad \tau_e^s = 0. \tag{38}$$

If furthermore the deterministic pollution tax rates, $\tau_k^{d*}$ and $\tau_e^{d*}$, are set optimally according to Equation (35), expected growth yields the first best level

$$\varphi^* = \varepsilon(A\eta^{*\alpha} - \eta^* - \beta). \tag{39}$$

Whether equilibrium growth increases or decreases due to the insurance against environmental risk again depends on the elasticity of intertemporal substitution which determines the dominance of income or substitution effects.

The corresponding first best abatement ratio is given by

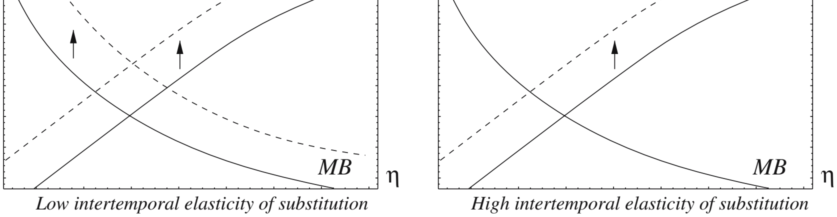
$$1 - \alpha A \eta^{*\alpha - 1} = \frac{\gamma}{\eta^*}(\varepsilon\beta + (1 - \varepsilon)(A\eta^{*\alpha} - \eta^*)). \tag{40}$$

On the one hand, due to the insurance, the volatility associated with environmental degradation decreases. A risk averse society gets ‘‘less afraid of’’ pollution and the optimal environmental quality level, $\eta^*$, decreases. In other words, the negative marginal cost of abatement due to the reduction of environmental risk vanishes. Therefore, marginal cost of abatement increases. This is the reason for the upward shift of the marginal cost curve (left hand side in equation 40) in Figure 3. $\eta^*$ ceteris paribus decreases.[18]

On the other hand, due to the insurance, the volatility associated with capital return diminishes. Hence, the certainty equivalent of capital return increases and in the empirically relevant case, $\varepsilon < 1$, optimal savings decrease. Less capital accumulation immediately leads to an increase in environmental quality. $\eta^*$ ceteris paribus increases. This is indicated with the upward shift of the marginal benefit curve (right hand side in equation 43) in Figure 3.

The over-all impact on optimal abatement is ambiguous and depends predominantly on the productive capacity of pollution, $\alpha$, which determines the magnitude of the increase in marginal costs, together with the elasticity of intertemporal substitution, $\varepsilon$, and environmental preferences, $\gamma$, which determine the magnitude of the increase in marginal benefits. Hence, the first best pollution level, $\eta^{*\hat{}-1}$, can increase or decrease due to the insurance against environmental risk.[19]

The magnitude of the upward shifts of the two curves is predominantly determined by intertemporal substitutability. The higher $\varepsilon$, the slighter is the




*Low intertemporal elasticity of substitution* ($\varepsilon < 1$). *High intertemporal elasticity of substitution* ($\varepsilon \to 1$).

*Figure 3.* Complete insurance and optimal abatement.

upward shift of marginal benefits, since the change in optimal savings is smaller. In contrast, marginal cost of abatement is independent from intertemporal substitutability. The limiting case $\varepsilon \to 1$ is displayed in Figure 3b. If the intertemporal elasticity of substitution is sufficiently high, the insurance induces an unambiguous increase in optimal pollution, $\eta^{*-1}$. In this case, the pollution increasing effect of the diminished riskiness associated with environmental productivity exceeds the growth increasing (pollution decreasing) effect of the diminished riskiness in capital return.

## 6. Conclusion

This paper analyzes efficient pollution taxation within a stochastic dynamic framework. Pollution is an inevitable by-product of production and reduces environmental quality. Moreover, there is an amenity value of environmental quality as well as a productivity enhancing effect. The latter is uncertain due to environmental risk: natural disasters like hurricanes or floods induce capital depreciation which is not known with certainty in advance. The Pareto-optimal steady state is described: the optimal abatement ratio depends positively on the amenity value and the productivity effect of environmental quality. Optimal abatement activity as well as optimal expected growth are affected in ambiguous way by uncertainty. The optimal adjustment of the savings decision on risk depends on the preferences, particularly on the elasticity of intertemporal substitution. It is shown that the riskiness of the environmental productivity effect may lead to a decrease in optimal environmental quality of a risk averse society, if capital accumulation is increased in order to provide an insurance against future income streams.

The individuals are assumed to perceive only part of their impact on pollution. Hence, there are externalities due to mis-perception, and the dynamic equilibrium is inefficient. It is shown that only the intratemporal decision between consumption and abatement is influenced. The intertemporal savings decision instead is independent from mis-perception.

In order to establish an efficient pollution tax, a general formulation of pollution taxation is introduced and the market equilibrium with arbitrarily given pollution taxation is determined. Two conditions are set up: An efficient pollution tax has to adjust the abatement effort to the optimal level and beyond has to be growth neutral. Steady state growth turns out to be inconsistent with a pollution tax which is levied directly on pollution. The resulting tax payment in this case is characterized by constant volatility, whereas income volatility increases through time. Hence, the relative riskiness changes and inhibits steady state growth. Instead, a linear pollution tax which is levied on capital income and abatement effort, can be used to internalize the externalities.

A first best solution, which additionally corrects for the institutional failure of absence of risk-pooling, can be obtained if the government uses the stochastic capital income tax in order to provide an insurance against environmental risk. With complete insurance, uncertainty vanishes. Nevertheless, optimal intertemporal savings as well as the optimal pollution level react ambiguously on the insurance, depending on intertemporal substitution. Hence, the optimal environmental quality may decrease due to insurance, since risk averse individuals get less afraid of the volatility of environmental risk.

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## Notes

1. The assumption of a general pollution function $P_i(t) = P(k_i(t), e_i(t))$ with constant and equal elasticities of pollution with respect to capital and abatement would end up with the same results.
2. As a consequence, individuals choose equal growth rates of capital and abatement in any equilibrium. Thus, any equilibrium will be sustainable due to constant pollution, even though pollution will be suboptimally high in equilibrium.
3. Instead of $p_i$, the stochastic process of productivity could also be defined by $dp_i$. In order to emphasize that pollution itself is a flow variable, I favor the notation $p_i$.
4. With the assumption of a Wiener process, I restrain the analysis to marginal shocks. Of course, various types of environmental quality would be characterized better with jumps in the pollution level. Nevertheless, Steger (2005) shows that the implications of marginal shocks (a Wiener process) and jumps (a Poisson process) are qualitatively the same. In order to keep the model simple, I assume environmental uncertainty to be marginal.
5. The results remain qualitatively unchanged for any constant population size $N$.
6. Nevertheless, a decrease in the volatility of environmental productivity due to lower environmental quality can be analyzed with the same model simply by setting $\alpha' < 0$.
7. Due to intergenerational altruism, the individuals can be interpreted as long-lived dynasties.
8. If an increase in pollution causes a decrease in the volatility of environmental productivity (if $\alpha' < 0$), the certainty equivalent of capital returns has to be positive in order to enable feasible solutions, that is, $-\rho\alpha/\alpha' A\eta^{-\alpha} \sigma^2 < 1$. This condition is equivalent to the requirement that uncertainty should not be too strong.
9. Only for the special case $\alpha = \alpha' = 0$, the solution is suggesting: optimal abatement in this case is given by $\eta = (\varepsilon\beta + (1-\varepsilon)A - \rho/2\ (1-\varepsilon)A^2\sigma^2)/(1/\gamma + 1 - \varepsilon)$ and optimal consumption results in $\mu = \gamma\,\eta$. This case is discussed with detail in Soretz (2003).
10. A detailed discussion is relocated to the appendix. I am indebted to an anonymous referee for this solution procedure.
11. For the distinction between risk premia (which depend on the intertemporal elasticity of substitution) and the motive for precautionary savings (which is based on risk aversion), see e.g. Kimball (1990), Weil (1993) or Gollier et al. (2000).

12. As already shown by Smith (1996a), positive consumption (feasibility) is neither necessary nor sufficient for the transversality condition to be satisfied. Instead, both conditions have to be verified separately. Nevertheless, Smith (1996a) proves that the feasibility condition as well as the transversality condition are met for empirically relevant parameterization: A relatively low intertemporal elasticity of substitution, $\varepsilon \leq 1$ is a sufficient condition for feasibility and a relatively high degree of risk aversion, $\rho \geq 1$, automatically satisfies the transversality condition. Subsequently, only parameter settings will be considered which satisfy the feasibility as well as the transversality condition.
13. See appendix for the derivation.
14. This outcome corresponds to the well-known growth neutrality of consumption tax.
15. In order to maintain consistency within individual optimization, I assume that individuals suppose the tax to depend on *perceived* pollution. Nevertheless, the argument is the same if the tax directly depends on "true" pollution.
16. The corresponding deterministic growth model is rapidly described by setting $\sigma^2 = 0$.
17. In this setting the Pareto-optimal consumption and abatement ratios result in $\mu = \frac{1}{1+\gamma(1-\varepsilon)}(\varepsilon\beta + (1-\varepsilon)A(1-\rho A\sigma^2/2))$ and $\eta = \gamma\mu$.
18. Gaube (2005) develops a related outcome in a deterministic framework: He shows that environmental quality may be higher in a second best situation with distortionary taxation than in a first best optimum with lump-sum taxes.
19. The case of decreasing marginal cost is not depicted with detail. Nevertheless, the results apply to both cases.

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## Appendix

## Determination of the social optimum

Existence and unicity of the social optimum are derived by the analysis of marginal cost, MC, and marginal benefit, MB, of abatement effort. According to condition (15), marginal cost and marginal benefit are given by

$$\begin{aligned} \mathrm{MC}(\eta) &= 1 - \alpha A \eta^{\alpha-1}\left(1 + \rho\frac{\alpha'}{\alpha}A\eta^{-2\alpha'-\alpha}\sigma^2\right) \\ \mathrm{MB}(\eta) &= \frac{\gamma}{\eta}\left(\varepsilon\beta + (1-\varepsilon)\left(A\eta^{\alpha} - \eta - \frac{\rho}{2}A^2\eta^{-2\alpha'}\sigma^2\right)\right). \end{aligned} \tag{A.1}$$

The thread is as follows: MB is unambiguously decreasing in abatement if the empirically relevant case $\varepsilon < 1$ applies. The slope of MC can be either positive or negative. Therefore, it is shown that MC does not decrease faster than MB. Moreover, MB is shown to be greater than MC for small enough values of $\eta$ and less than MB at the upper bound of feasible $\eta$. These arguments together give existence and unicity of the optimal abatement ratio.

The slope of marginal benefit evolves to

$$\begin{aligned} \mathrm{MB}' \equiv \frac{\partial \mathrm{MB}}{\partial \eta} &= -\frac{\gamma\mu}{\eta^2} - \frac{\gamma}{\eta}(1-\varepsilon)\left(1 - \alpha A\eta^{\alpha-1}\left(1 + \rho\frac{\alpha'}{\alpha}A\eta^{-2\alpha'-\alpha}\sigma^2\right)\right) \\ &= -\frac{\mathrm{MB}}{\eta} - \frac{\gamma}{\eta}(1-\varepsilon)\mathrm{MC}. \end{aligned} \tag{A.2}$$

In the neighborhood of the intersection between marginal benefit and marginal cost, MB can be approximated by MC. Hence, the slope of MB in the intersection with MC is given by

$$\mathrm{MB}' = -\frac{\mathrm{MC}}{\eta}(1+\gamma(1-\varepsilon)). \tag{A.3}$$

In the empirically relevant case, $\varepsilon < 1$, marginal benefit decreases with a rise in abatement, $\mathrm{MB}' < 0$.

Nevertheless, the slope of MC is ambiguous, as can be seen with

$$\mathrm{MC}' \equiv \frac{\partial \mathrm{MC}}{\partial \eta} = -\alpha(\alpha-1)A\eta^{\alpha-2} + \alpha'(2\alpha'+1)\rho A^2\eta^{-2\alpha'-2}\sigma^2 \gtrless 0 \tag{A.4}$$

$$\Longleftrightarrow \qquad \eta \lessgtr \left(\frac{\alpha(\alpha-1)}{\alpha'(2\alpha+1)\rho A\sigma^2}\right)^{-\frac{1}{2\alpha'+\alpha}}. \tag{A.5}$$

For small values of abatement, MC increases, for large values, it decreases. Only for $\alpha \leq 1$, the outcome is clear cut: (A.4) shows that $\mathrm{MC}' > 0 \; \forall \; \eta > 0$. In the case $\alpha > 1$ it is not possible to exclude the maximum of MC from the range of feasible abatement effort: An abatement ratio is feasible if it is positive and below expected capital productivity, since abatement effort cannot exceed average income. Hence, $0 < \eta < A^{\frac{1}{1+\alpha}}$ describes the range of feasible abatement effort, and the maximum given in Equation (A.5) may well be situated within this interval.

Hence, to prove existence and unicity of optimal abatement, it will be shown that around any intersection of MB and MC, the slope of MC is greater than that of MB, that is, $\mathrm{MC}' > \mathrm{MB}'$. To show this relation, the second order necessary condition with respect to $e$ has to be determined

$$\begin{aligned} \frac{\partial^2 \mathcal{B}}{\partial e^2} &\Longrightarrow \gamma(\gamma(1-1/\varepsilon)-1)\mu\eta^{-2} + \alpha(\alpha-1)A\eta^{-\alpha-2} \\ &- \rho A^2\alpha'(2\alpha'+1)\eta^{-2\alpha'-2}\sigma^2 \overset{!}{<} 0 \\ &\Longleftrightarrow \quad \gamma(\gamma(1-1/\varepsilon)-1)\frac{\eta}{\gamma}\mathrm{MC}\eta^{-2} - \mathrm{MC}' < 0. \end{aligned} \tag{A.6}$$

From Equation (A.3) follows immediately $\mathrm{MC} = \mathrm{MB}'\eta/(1+\gamma(1-\varepsilon))$ which is valid in the neighborhood of optimal abatement, and therefore

$$\frac{\gamma(1-1/\varepsilon)-1}{1+\gamma(1-\varepsilon)}\mathrm{MB}' + \mathrm{MC}' > 0 \Longleftrightarrow \frac{1+1/\varepsilon\gamma(1-\varepsilon)}{1+\gamma(1-\varepsilon)}\mathrm{MB}' < \mathrm{MC}'. \tag{A.7}$$

The fraction in Equation (A.7) is greater than one as long as $\varepsilon$ is below unity. Hence, the second order condition (A.7) is sufficient for $\mathrm{MB}'$ to be less than $\mathrm{MC}'$.

The remaining points are to verify that *MB* is greater (smaller) than MC for small (large) values of $\eta$. At the upper bound of feasible abatement effort, $\bar{\eta} \equiv A^{\frac{1}{1+\alpha}}$, expected growth as well as consumption vanish, $\varphi(\bar{\eta}) = -\mu(\bar{\eta}) = 0$, and therefore

$$\mathrm{MB}(\bar{\eta}) = \frac{\gamma}{\bar{\eta}}\mu(\bar{\eta}) = 0. \tag{A.8}$$

Due to $\mathrm{MC}(\bar{\eta}) > 0$, marginal cost is greater than marginal benefit at this upper bound of abatement activity.

With abatement activity sufficiently low, MC gets negative

$$\lim_{\eta\to 0} \mathrm{MC} = \lim_{\eta\to 0} 1 - \alpha A\eta^{\alpha-1}\left(1+\rho\frac{\alpha'}{\alpha}A\eta^{-2\alpha'-\alpha}\sigma^2\right) = -\infty. \tag{A.9}$$

From MB > 0 ∀ $\eta$, it is obvious that marginal benefit is positive. Therefore, marginal benefit is greater than marginal cost at the lower bound of abatement activity. Consequently, there has to be exactly one intersection between marginal cost and marginal benefit, which is situated in the feasible range and indicates socially optimal abatement effort. The argument is summarized in Figure 1 in Section 3.

## Market equilibrium

Dynamic market equilibrium is determined by the maximization of the stochastic Bellman equation

$$\begin{aligned}\mathcal{B} &= \frac{1-\rho}{1-1/\varepsilon}(c_i P_{\mathrm{p}}^{-\gamma})^{1-1/\varepsilon} - \beta G((1-\rho)J_i(k_i)) \\ &\quad + (1-\rho)G'((1-\rho)J_i(k_i))\left(J_i'(k_i)\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}J_i''(k_i)\sigma_{k_i}^2\right)\end{aligned} \tag{A.10}$$

with the variance of individual capital, $\sigma_{k_i}^2 = A^2 k_i^2 P_{\mathrm{p}}^{2\alpha'} \sigma^2$. The derivation of the Bellman equation with respect to consumption remains unchanged as in (8) and together with the conjecture of constant and equal consumption and abatement ratios, μ and $\eta$, yields the same guess for the value function (10).

The first order conditions with respect to abatement effort and capital accumulation now have to account for perceived pollution:

$$\gamma(1-\delta)(c_i P_{\mathrm{p}}^{-\gamma})^{1-1/\varepsilon} + G_i'\left(J_i'(\alpha(1-\delta)Ak_iP_{\mathrm{p}}^{-\alpha} - e_i) + \frac{1}{2}J_i''e_i\frac{\partial\sigma_{k_i}^2}{\partial e_i}\right) \stackrel{!}{=} 0 \tag{A.11}$$

$$\begin{aligned}&-\gamma(1-\delta)(cP_{\mathrm{p}}^{-\gamma})^{1-1/\varepsilon}k_i^{-1} + (1-\rho)G_i''J_i'\left(J_i'\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}J_i''\sigma_{k_i}^2\right) + \\ &+ G_i'\left(J_i'(AP_{\mathrm{p}}^{-\alpha}(1-\alpha(1-\delta)) - \beta) + J_i''\left(\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}\frac{\partial\sigma_{k_i}^2}{\partial k_i}\right) + \frac{1}{2}J_i'''\sigma_{k_i}^2\right) \stackrel{!}{=} 0.\end{aligned} \tag{A.12}$$

Transformation of these conditions follows the same procedure as described above with the Pareto-Optimum. Only the partial derivatives of perceived pollution change.

With respect to the dynamic market equilibrium with pollution tax, the Bellman equation remains unchanged as given in Equation (A.10) for the dynamic market equilibrium. Again, the derivative with respect to consumption is given in (8) and together with the conjecture of constant and equal growth rates (9) leads to the same guess of the value function as derived in (10).

The derivatives with respect to individual abatement expenditures and capital accumulation result in

$$\gamma(1-\delta)(c_i P_{\mathrm{p}}^{-\gamma})^{1-1/\varepsilon} + G_i'\left(J_i'(\alpha(1-\delta)Ak_iP_{\mathrm{p}}^{-\alpha} - e_i(1+T_{\mathrm{e}_i}^{\mathrm{d}})) + \frac{1}{2}J_i''e_i\frac{\partial\sigma_{k_i}^2}{\partial e_i}\right) \stackrel{!}{=} 0 \tag{A.13}$$

$$-\gamma(1-\delta)(cP_{\mathrm{p}}^{-\gamma})^{1-1/\varepsilon}k^{-1} + (1-\rho)G_i''J_i'\left(J_i'\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}J_i''\sigma_{k_i}^2\right) + \tag{A.14}$$

$$+G_i'\left(J_i'(AP_{\mathrm{p}}^{-\alpha}(1-\alpha(1-\delta)) - T_{k_i}^{\mathrm{d}} - \beta) + J_i''\left(\frac{\mathrm{E}[dk_i]}{dt} + \frac{1}{2}\frac{\partial\sigma_{k_i}^2}{\partial k_i}\right) + \frac{1}{2}J_i'''\sigma_{k_i}^2\right) \stackrel{!}{=} 0$$

with the respective derivatives of the variance of capital given by

$$\frac{\partial\sigma_{k_i}^2}{\partial e_i} = -2\sigma^2\eta^{-1}(A\eta^{-\alpha'}k_i - T^{\mathrm{s}})(\alpha'(1-\delta)A\eta^{-\alpha'} - \eta T_{e_i}^{\mathrm{s}}) \tag{A.15}$$

$$\frac{\partial\sigma_{k_i}^2}{\partial k_i} = 2\sigma^2(A\eta^{-\alpha'}k_i - T^{\mathrm{s}})((1+\alpha'(1-\delta))A\eta^{-\alpha'} - T_{k_i}^{\mathrm{s}}). \tag{A.16}$$